SNAP 2017. Laplace's equation and conformal maps.

Problem Set 1

1. Consider the holomorphic map $w = z^2$. For $a, b > 0$, what are the images of the hyperbolae
$$x^2 - y^2 = a, \quad 2xy = b$$
under this map? (We are writing $z = x + iy$).

   Solution. $w = (x^2 - y^2) + 2xyi$ so writing $w = u + iv$ the images are the lines $u = a$ and $v = b$.

2. Let a, b be nonzero real numbers. Show that the map $w = 1/z$ transforms the vertical line $x = a$ to a circle through the origin centered at $(\frac{1}{2a}, 0)$. What about horizontal lines $y = b$?

   Solution. When $x = a$ we have $w = u + iv$ with $u = a/(a^2 + y^2)$ and $v = -y/(a^2 + y^2)$ and a straightforward calculation shows that $(u - 1/2a)^2 + v^2 = 1/(4a^2)$. A similar calculation shows that horizontal lines $y = b$ get sent to circles through the origin with center $(0, -1/2b)$.

3. Recall that the holomorphic function $\sin z$ is defined by $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$. Recall also that $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$.

   (a) Show that $\sin z = \sin x \cosh y + i \cos x \sinh y$, where we recall that hyperbolic sine and cosine are defined by
$$\sinh y = \frac{e^y - e^{-y}}{2}, \quad \cosh y = \frac{e^y + e^{-y}}{2}.$$

   Solution. Straightforward calculation.

   (b) For $a, b \neq 0$, show that the lines $x = a$ and $y = b$ are transformed into hyperbolae and ellipses respectively, and find their equations. (Hint: use the identities $\cosh^2 x - \sinh^2 x \equiv 1$ and $\sin^2 x + \cos^2 x \equiv 1$.)

   Solution. $x = a$ becomes the hyperbola $\frac{u^2}{\sin^2 a} - \frac{v^2}{\cos^2 a} = 1$ and $y = b$ becomes the ellipse $\frac{u^2}{\cosh^2 b} + \frac{v^2}{\sinh^2 b} = 1$.

4. Find a conformal map from the upper half plane $\{y > 0\}$ onto $\{w \in \mathbb{C} \mid |w| > 1\}$.

   Solution. $w = e^{-iz}$.

5. Find the Möbius transformation $f(z) = \dfrac{az + b}{cz + d}$ which maps 1 to 1, -1 to -1 and i to 0.

   Solution. $f(z) = \frac{i(z-i)}{z+i}$.

6. Without computing the map explicitly, find the image of the open unit disk $\{|z| < 1\}$ under the Möbius transformation that maps -1 to $-i$, 1 to $2i$ and i to 0.

   Solution. The unit circle in the clockwise direction maps to the imaginary axis in the direction of positive y, so the image of the open unit disk is the right half plane.

7. (a) Find a holomorphic surjection from the upper half plane onto $\mathbb{C}$.
   Solution. Apply $z \mapsto z - 1$ then $z \mapsto z^2$.

   (b) Why is there no holomorphic surjection from $\mathbb{C}$ onto the upper half plane? *(Hint: Liouville's Theorem says that a bounded holomorphic map on $\mathbb{C}$ must be constant.)*
   Solution. If one existed, then combine with a holomorphic map from the upper half plane to the unit disc to get a contradiction to Liouville.

8. Consider the Joukowsky map $f(z) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$.

   (a) Where is $f(z)$ conformal? (Recall that a map $g(z)$ is *conformal* at z if it is holomorphic at z and $g'(z) \neq 0$.)
   Solution. For all z except ± 1 and 0.

   (b) Show that for $w \in \mathbb{C}$, the equation $f(z) = w$ has exactly two solutions z except when $w = \pm 1$.
   Solution. $f(z) = w$ becomes the quadratic equation $z^2 - 2zw + 1 = 0$ which by the quadratic formula has two distinct solutions except when $w = \pm 1$. In fact, notice that if z solves $f(z) = w$ then so does $1/z$, and these are distinct except when $w = \pm 1$.

   (c) Show that $w = f(z)$ is a bijective conformal map from the half disk $\{|z| < 1,\ \operatorname{Im} z > 0\}$ to the lower half plane $\{w \in \mathbb{C} \mid \operatorname{Im} w < 0\}$.
   Solution. By (a), f is conformal. For z with $|z| < 1$ we have $\operatorname{Im} f(z) = \frac{1}{2}(y - y/(x^2+y^2))$ so $\operatorname{Im} f(z) < 0$ when $y > 0$ and $\operatorname{Im} f(z) > 0$ when $y < 0$. In particular, f maps the half disk $\{|z| < 1,\ \operatorname{Im} z > 0\}$ to the lower half plane. Given any w in the lower half plane, by (b) there are two solutions to $f(z) = w$, which we can write as z and $1/z$. Exactly of these lies in the half disk $\{|z| < 1,\ \operatorname{Im} z > 0\}$.

9. Using the Joukowsky map, composed with other conformal maps, find the image of the semi-infinite vertical strip $S = \{(x, y) \mid |x| < \pi/2,\ y > 0\}$ under the map $w = \sin z$. What are the images of the horizontal line segments $y = c$ inside S under the map?

   Solution. Consider the map $z \mapsto -ie^{iz}$ composed with the Joukowsky map to get $\sin z$. The image is the upper half plane. The horizontal line segments $y = c$ map to the top half of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with $a = \frac{1}{2}(e^c + e^{-c})$ and $b = \frac{1}{2}(e^c - e^{-c})$.